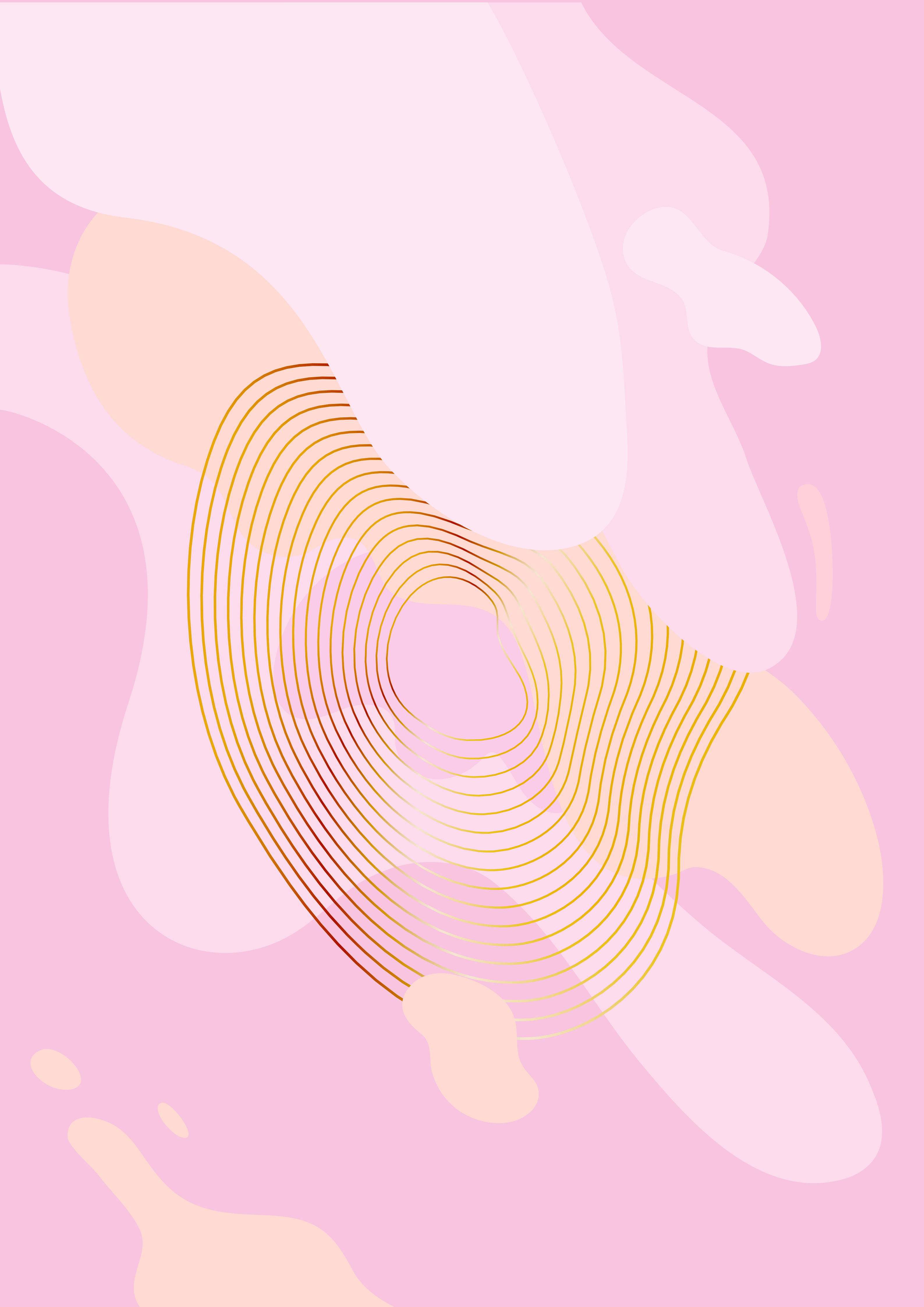

WEEK 1:

Theorem (Scholze): If X is a perfectoid space then $\overset{\text{mixed char}/\mathbb{Q}_p}{X_{\acute{e}t}} \cong \overset{\text{tilt of } X \text{, char } p}{X^{\flat}_{\acute{e}t}}$

Usual examples:

* Y quasi-proj $/\mathbb{Q}_p\ (\mathbb{Z}_p) \xrightarrow{\text{obtain}} X$ perfectoid. $(*)$

Goal 1: How to apply the thm. as in $(*)$

Goal 2: Understand the notions and possibly the proof of the thm.

Definition: X is a perfectoid space if it is an adic-space and locally $\cong \mathrm{Spa}(A, A^+)$ where (A, A^+) is a perfectoid Huber pair.

(Spa: variant of Spec for adic spaces)

Definition: (A, A^+) is a perfectoid Huber pair if it is

* A is a perfectoid Tate ring
* A^+ a ring of integral elements in A. (open normal subring $\subseteq A^\circ$)

Definition: Let B be a topological ring.

(a) B is non-archimedean if $\exists$ a fundamental system of open neigh. $\{B_i\}$ of $0 \in B$ st B_i are additive subgroups.

(b) B is adic if $B_i = I^i$ for some $I \trianglelefteq B$.

Examples 1:

* $k[x]$ $I = (x)$
* $k[\![x]\!]$ $I = (x)$
* $\mathbb{Z}_p$ $I = (p)$ — DVR, p a parameter.

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z} \to (0, p, p, -) = p$

(c) B is f-adic if

(i) there exists $B_0 \subseteq B$ open subring

(ii) $I \subseteq B_0$ finitely gen. with which B_0 is I-adic.

Examples 2:

* prev. examples w/ $B_0 = B$
* $B = \mathbb{Q}_p$ $B_0 = \mathbb{Z}_p$
* $B = k(\!(x)\!)$ $B_0 = k[\![x]\!]$.

(d) B is a Tate ring if B is f-adic + there is a topologically nilpotent in B.

$b \in B$ is top. nilpotent if $\lim_{n \to \infty} b^n \to 0$.

Examples:

* Examples 1 not Tate
* Examples 2 Tate.

Valuations: (ring w/ multiplicative valution = normed ring).

additive	multiplicative
	take $\lvert\cdot\rvert = e^{-v(\cdot)}$ and $\lvert\cdot\rvert_p = p^{-v(\cdot)}$
• $v(a \cdot b) = v(a) + v(b)$	• $\lvert a \cdot b\rvert = \lvert a\rvert \cdot \lvert b\rvert$
• $v(a+b) \geq \min(v(a), v(b))$	• $\lvert a+b\rvert \leq \max\{\lvert a\rvert, \lvert b\rvert\}$
• $v(1) = 0$	• $\lvert 1\rvert = 1$ • $\lvert 0\rvert = 0$
• $v(0) = \infty$	

Example: $(k, \lvert\cdot\rvert)$ normed field,

Tate algebra.

$$k\langle x_1, -, x_d\rangle = \{\textstyle\sum_{\nu \in \mathbb{N}^d} a_\nu x^\nu \in k[\![x_1, -, x_d]\!] \mid \max\{\lvert a_\nu\rvert \mid \nu = \deg n\} \xrightarrow{n \to \infty} 0\}.$$

e.g. $\sum_{n \in \mathbb{N}} p^n x^n \in \mathbb{Q}_p\langle x\rangle$ but $\mathbb{Q}_p\langle x\rangle \not\ni \sum_{n \in \mathbb{N}} x^n = 1/x-1$.

* $\sum_{i \in \mathbb{N}} a_i$ $a_i \in k$, $\lvert\sum_{i=r}^{s} a_i\rvert \leq \max\{\lvert a_i\rvert \mid i = r, -, s\} \Rightarrow \sum_{i \in \mathbb{N}} a_i$ is conv. $\Leftrightarrow \lvert a_i\rvert \to 0$.

$\Rightarrow k\langle x_1, -, x_d\rangle = \{\sum a_\nu x^\nu \mid$ convergent on the closed unit disc$\}$.

norm on the Tate algebra: $|\sum a_\nu x^\nu| = \max|a_\nu|$ except for $|0|=0$.

Homework 1: $k\langle x_1,\dots,x_n\rangle$ is a normed ring w/ this norm.
Homework 2: for $f\in k\langle x_1,\dots,x_n\rangle$ is it true $|f| = \max\{|f(c)| \mid c\in D\}$
maybe condition on k. What if sup instead of max.

Note: $\widehat{\mathbb{Z}_p[x_1,\dots,x_n]}^p[1/p] \cong \mathbb{Q}_p\langle x_1,\dots,x_n\rangle$
in AG:

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{Z}_p} \arrow[d] & \rightsquigarrow & \hat{\mathbb{A}}^1_{\mathbb{Z}_p} \arrow[d] & ???\ \mathbb{Q}_p\langle x_1,\dots,x_n\rangle \arrow[l, "\text{generic fiber}"'] \\
\operatorname{Spec}\mathbb{Z}_p & & \operatorname{Spec}\mathbb{Z}_p & \operatorname{Spec}\mathbb{Q}_p \arrow[l]
\end{tikzcd}$$

Boundedness in topological rings:

Definition: B topo. ring.
(a) $S\subseteq B$ is bounded if $\forall\, 0\in U$ open $\exists\, 0\in V$ open st $S.V\subseteq U$.
(b) $S\subseteq B$ is power bounded if $\bigcup_{n\geq 0} S^n$ is bounded.

Example: B normed ring, $U_c := \{b\in B \mid |b|<c\}$ open by definition but also,
$D_c = \{b\in B \mid |b|\leq c\}$ is open. Both are bounded.
Take some open U as in def $\Rightarrow \exists U_d \subseteq U$ take $V = U_{d/c}$.
$\Rightarrow D_1$ contains only power bounded elements.
Take $b\in B\setminus D_1 \Rightarrow \{1, b, b^2, \dots\}$ is it bounded? Take $U=U_1$ as in definition
assume $\exists V$ as in the definition $\Rightarrow$ no U_ε is contained in $V \Rightarrow$ contradiction.
So:
$D_1 = \{$power bounded elts.$\}$. $\#$

Definition: A top ring $\Rightarrow A^\circ = \{$power bounded elts$\}$, $A^{\circ\circ} = \{$topologically nilpotent elts$\}$
Homework 3: (i) top nilp $\Rightarrow$ power bounded
(ii) $A^\circ \subseteq A$ subring (is it open?) (iii) $A^{\circ\circ} \trianglelefteq A^\circ$ (is it prime?)

Example: B normed ring, $B^\circ = \{b\in B \mid |b|\leq 1\}$, $B^{\circ\circ} = \{b\in B \mid |b|<1\}$.
prime ideal.

$$\begin{array}{ccccc} B & \supseteq & B^\circ & \supseteq & B^{\circ\circ} \\ \| & & \| & & \| \\ \mathbb{Q}_p & \supseteq & \mathbb{Z}_p & \supseteq & p\mathbb{Z}_p \end{array}$$

Definition: Elements of $A^{\circ\circ}\setminus\{0\}$ are pseudo-uniformizers.

Definition: A is a non-arch. topological ring. A is called uniform if A° is bounded.
Example: normed rings.
Definition: Tate ring A is perfectoid if it is (p prime fixed)
- Complete
- uniform
- $\exists\, \varpi\in A$ pseudo uniformizer (unit in A) such that
  (i) $\varpi^p \mid p$ (ii) $A^\circ/\varpi \longrightarrow A^\circ/\varpi^p$ is an isomorphism.
  $x \longmapsto x^p$

Note: perfectoid Tate ring is not a perfectoid ring but A° is.

WEEK 2

$$\begin{tikzcd} \overset{A}{X} \arrow[d, "\text{ft. type}"'] & X_{\mathbb{Z}_p} \arrow[l] \arrow[d] \\ \operatorname{Spec}\mathbb{Z} & \operatorname{Spec}\mathbb{Z}_p \arrow[l] \end{tikzcd}$$

$A_{(p)} \subseteq A\otimes_{\mathbb{Z}}\mathbb{Z}_p \subseteq \widehat{A\otimes_{\mathbb{Z}}\mathbb{Z}_p}$

Ex:
$$\begin{tikzcd} \mathbb{Z}[t] \arrow[d] & \mathbb{Z}_p[t] \arrow[l] \arrow[d] \subseteq \widehat{\mathbb{Z}_p[t]}^p = \mathbb{Z}_p\langle t\rangle \\ \mathbb{Z} & \mathbb{Z}_p \arrow[l] \end{tikzcd}$$

$\mathbb{Z}_p\langle t\rangle \cap \ldots$: $\widehat{\mathbb{Z}_p[t]}^p[1/p] \cong \mathbb{Q}_p\langle t\rangle$.

Think these examples through!

Examples of perfectoid Tate rings:

(i) Is $\mathbb{Q}_p$ perfectoid Tate? $A^\circ = \mathbb{Z}_p \ni p$ irreducible so $\varpi \in A^{\circ\circ} = p\mathbb{Z}_p$ and ϖ^p doesn't divide p. NO

(ii) $\widehat{\mathbb{Q}_p[p^{1/p^\infty}]}^p$

elements are of the form $\lim_{i\to\infty} c_i$ st $|c_i - c_{i+1}|_p \xrightarrow{i\to\infty} 0$. One can show that

$|\lim_{i\to\infty} c_i| \leq 1 \Rightarrow \forall n \gg 0 \quad |c_n| \leq 1. \Rightarrow A^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}^p$ & $A^{\circ\circ} = (p, p^{1/p}, \_) = (p^{1/p^\infty})$
take $\varpi = p^{1/p}$ then

$$A_0/p \cong \mathbb{F}_p[x_1, x_2, \_]/_{(x_1^p, x_2^p = x_1, \_)} \cong \mathbb{F}_p[t^{1/p^\infty}]/t$$

$$A_0/p^{1/p} \cong \mathbb{F}_p[x_1, x_2, \_]/t^{1/p} \quad \& \quad A_0/p^{1/p} \xrightarrow{\cdot p} A_0/p \text{ is bijective.}$$

Proposition: A_0 is normal for any normal uniform non-archd. ring.

Proof: $a \in \operatorname{Frac} A$, $a^n + \sum_i b_i a^i = 0 \quad b_i \in A_0$.
$\Rightarrow \{a^i \mid i \geq 0\} \in A^\circ + A^\circ . a + \_ + A^\circ . a^{n-1} = S$ need to show S is bounded.
Fix $0 \in U$ open $\subseteq A$ need $V \ni 0$ open st $V.S \subseteq U$.
by non-archd. can suppose U to be an additive subgroup.
Then it is enough to show that $V . A^\circ . a_i \subseteq U$ for $0 \leq i \leq n-1$.

A° bounded $\Rightarrow \exists W \ni 0$ open st $A^\circ . W \subseteq U$. Take $V = \bigcap_{i=0}^{n-1} (W . \frac{1}{a_i} \cap A^\circ)$.

Adic spaces:

What is $\operatorname{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x\rangle)$?

Definition: X adic-space is $(X, \mathcal{O}_X, |\cdot|_x \ \forall x \in X)$ $\mathcal{O}_X$ presheaf, $|\cdot|_x$ valuations such that locally $X \cong \operatorname{Spa}(A, A^+)$.
$\rightarrow A^+ \subseteq A^\circ$ int. closed + open

Valuations: $R \to \Gamma \cup \{0\}$ $\quad \Gamma$ totally ordered abelian group 1 identity element.
Examples: $*\ \Gamma = (\mathbb{R}_{>0}, \cdot)$ $\quad *\ \Gamma = \mathbb{Z}\times\mathbb{Z}$ with lexicographic order.

The valuation $|\cdot|_x$ is on $\mathcal{O}_{X,x}$ and $\operatorname{Supp}|\cdot|_x := \{a \in \mathcal{O}_{X,x} \mid |a|_x = 0\} = m_{X,x}$.

Definition: Morphisms of adic spaces: $(f, f^\#)$ st $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ morphisms of presheaves of top. rings st $|\cdot|_{f(x)} \circ f_x^\# = |\cdot|_x$.

$\operatorname{Spa}(A, A^+)$: continuous valuations $\quad$ valuations (identify those with $\Gamma \cong \Gamma'$ & suppose $\operatorname{im}(|\cdot|)$ gen. Γ)
points: $\{|\cdot|_x \in \operatorname{Cont}(A) \mid |A^+|_x \leq 1\}$ $\quad \operatorname{Cont}(A) = \{x \in \operatorname{Spv}(A) \mid \forall \gamma \in \Gamma \cdot \{a \in A : |a| < \gamma\}$ is open$\}$.

Note: $\operatorname{Supp}|\cdot|_x \subseteq A$ is a prime ideal. $\quad R_x := \{a \in A \mid |a| \leq 1\}$ subring.
$P_x := \{a \in A \mid |a| < 1\} \subseteq R_x$ prime.

Proposition: $\{\text{val on } A \text{ with } Supp = \{0\}\} \longleftrightarrow \{\text{val. on } K(A)\}$.

Example: Valuations of $\mathbb{Q}$: $\leftrightarrow$ valuations of $\mathbb{Z}$ with $Supp = 0$. Let $|\cdot|$ be such a valuation. $R_{|\cdot|} = \mathbb{Z}$ $\quad P_{|\cdot|} = (0)$ or $p\mathbb{Z}$.
if $P_{|\cdot|} = (0) \Rightarrow |\cdot| = |\cdot|_1$ $\quad$ if $P_{|\cdot|} = p\mathbb{Z}. \Rightarrow \forall n \in \mathbb{Z}\ n = p^i \cdot m \quad m \notin p\mathbb{Z}$.
$|n| = |p|^i \Rightarrow |\cdot| = |\cdot|_p$.

topology: main idea: $\{x \in Spa(A, A^+) \mid |\frac{f}{g}|_x \leq 1\} \longrightarrow$ precise def. $R\left(\frac{f_1, \ldots, f_n}{g}\right)$
where $(f_1, \ldots, f_n) \subseteq_{open} A$ and $R\left(\frac{f_1 \ldots f_n}{g}\right) = \{x \in Spa(A, A^+) \mid \forall i\ |f_i| \leq |g| \neq 0\}$.

Example: $\lambda \in \mathbb{D}_{\mathbb{Q}_p} = \mathbb{Z}_p$ $\quad \mathbb{Q}_p\langle X\rangle \ni \sum a_i x^i \longmapsto \sum a_i \lambda^i$

$$R\left(\frac{X}{p}\right) = \{|\cdot|_\lambda \mid |X|_\lambda \leq |p|_\lambda\} = p\mathbb{Z}_p.$$

WEEK 3:

Definition: Valuation subring, $R_{|\cdot|_x} = \{a \in A \mid |a|_x \leq 1\}$, $P_{|\cdot|_x} = \{a \in A \mid |a|_x < 1\}$ is a prime ideal of $R_{|\cdot|_x}$ and $Supp|\cdot|_x = \{a \in A \mid |a|_x = 0\}$ is a prime ideal of A.

Definition: Let K be a field. $A \subseteq K$ is a valution ring if $\forall x \in K$ non-zero either $x \in A$ or $x^{-1} \in A$. There is a correspondence:
$$R \subseteq K \text{ val ring} \iff |\cdot|_x : K \to \Gamma_x \text{ valuation}.$$

Short term goal: Computing $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$ & $Spa(\mathbb{Z}_p, \mathbb{Z}_p)$.

Proposition: $|\cdot|_x \in Cont(A) \Rightarrow D_c = \{a \in A \mid |a| \leq c\}$ open & closed in A $\forall c \in \Gamma_x$.
Claim: For any $|\cdot| \in Spv(A)$, $|a| < |b| \Rightarrow |a+b| = |b|$.
Proof: $|a+b| \leq \max\{|a|, |b|\} \Rightarrow |a+b| \leq |b|$.
$|b| \leq \max\{|-a|, |b+a|\} \Rightarrow |b| \leq |b+a|$.
Proof: Let $a \in A$ st $|a|_x \leq c$. $B(a,c) = \{b \in A \mid |b-a|_x < c\}$ This is open in A because $|\cdot|_x$ is continuous. Strict triangle inequality implies that D_c is open.
Claim $\Rightarrow \forall a \in A \setminus D_c$ then $B(a,c) \subseteq A \setminus D_c$.

Remark: Let $|\cdot|_x$ be a cont. valuation on $\mathbb{Z}_p$ or $\mathbb{Q}_p$ then $R_x \subseteq_{closed} \mathbb{Q}_p$ or $\mathbb{Z}_p$ but $\mathbb{Z} \subseteq R_x \Rightarrow \overline{\mathbb{Z}} \subseteq R_x \Rightarrow \mathbb{Z}_p \subseteq R_x$.

Example: Let $|\cdot|_x \in Cont(\mathbb{Q}_p)$. $\mathbb{Q}_p$ is a field $\Rightarrow Supp|\cdot|_x = 0$. As $|\cdot|_1 \notin Cont(\mathbb{Q}_p)$,
$\Rightarrow \mathbb{Z}_p \subseteq R_x \neq \mathbb{Q}_p \Rightarrow \mathbb{Z}_p = R_x$. (there is no ring between $\mathbb{Z}_p$ and $\mathbb{Q}_p$).
$\Rightarrow P_x = p\mathbb{Z}_p \Rightarrow |\cdot|_x = |\cdot|_p \Rightarrow Cont(\mathbb{Q}_p) = \{|\cdot|_p\}$.
This shows that $Spa(\mathbb{Q}_p, \mathbb{Z}_p) = \{|\cdot|_p\}$.

Example: $Spa(\mathbb{Z}_p, \mathbb{Z}_p)$, previous argument shows that if $Supp|\cdot| = 0$ then $|\cdot| = |\cdot|_p$. Suppose $Supp|\cdot|_x = p\mathbb{Z}_p \Rightarrow |\cdot|_x = |\cdot|_1 \circ ev_p$ where $ev_p : \mathbb{Z}_p \to \mathbb{F}_p$.
$\Rightarrow Spa(\mathbb{Z}_p, \mathbb{Z}_p) = \{|\cdot|_p, |\cdot|_1 \circ ev =: |\cdot|_{1,p}\}$
What is the topology on $Spa(\mathbb{Z}_p, \mathbb{Z}_p)$? $\quad |\cdot|_{1,p} \in R\left(\frac{f_1 \ldots f_n}{g}\right) \Rightarrow |g|_{1,p} \neq 0$
$\Rightarrow g \notin p\mathbb{Z}_p$.
$\Rightarrow |g|_p = 1 \Rightarrow |\cdot|_p \in R\left(\frac{f_1 \ldots f_n}{g}\right)$. To conclude suffices to show that $\{|\cdot|_p\}$ is open but $R\left(\frac{p}{p}\right) = \{|\cdot|_p\}$

Midterm Goal: Understanding $\mathbb{C}_p\langle x\rangle$, $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}^{p}$.
Shortterm Goal: algebra of $k\langle x_1,\dots,x_n\rangle$ for k normed + complete field.

Homework 2 (Domenico's / Correct version): For $f\in k\langle x_1,\dots,x_n\rangle$
$|f| = \max\{|f(a)| : a\in \mathbb{D}(\bar{k})\}$ **Hint:** if $|f|=1$ then the valuation ring is $R_{|\cdot|}\langle x_1,\dots,x_n\rangle$ where $R_{|\cdot|}$ is the valuation ring of k.

$$\text{Then } R_{|\cdot|}\langle x_1,\dots,x_n\rangle / p.R_{|\cdot|}\langle x_1,\dots,x_n\rangle \cong (R/p)[x_1,\dots,x_n]$$
$$f \longmapsto \bar{f} \neq 0.$$

Homework 4: $f\in k\langle x_1,\dots,x_n\rangle$ w/ $|f|=1$ then f is invertible $\Leftrightarrow$ as above $R_{|\cdot|}\langle x_1,\dots,x_n\rangle / p.R_{|\cdot|}\langle x_1,\dots,x_n\rangle \cong (R/p)[x_1,\dots,x_n]$, $f\longmapsto \bar{f}$ and $\bar{f}\in (R/p)^\times$

$\Updownarrow$
|constant coeff| $=1$
|all other coeff| <1.

Very Short term goal: Weierstrass division $\leftarrow$ works only for distinguished elements.

Definition: $g\in k\langle x_1,\dots,x_n\rangle$ is x_n-distinguished of order s if when you write $g=\sum_{i\in\mathbb{N}} g_i x_n^i$ where $g_i\in k\langle x_1,\dots,x_{n-1}\rangle$ then
(1) $g_s\in k\langle x_1,\dots,x_{n-1}\rangle^\times$ (2) $|g_i|<|g_s|$ $\forall i>s$.

Remark: $0\neq f\in k\langle x\rangle \Rightarrow f$ is x-distinguished.

Lemma: $f_1,\dots,f_r\in k\langle x_1,\dots,x_n\rangle \Rightarrow \exists a_1,\dots,a_{n-1}\in\mathbb{N}$ st the continuous aut:
$$\sigma: k\langle x_1,\dots,x_n\rangle \to k\langle x_1,\dots,x_n\rangle$$
$$x_i \longmapsto x_i + x_n^{a_i}$$
$$x_n \longmapsto x_n$$
st $\sigma(f_i)$ is x_n-distinguished. Moreover, σ preserves $|\cdot|$.

Example: $xy\in k\langle x,y\rangle$ is not distinguished. Consider $(x+y^a)y = xy+y^{1+a}$ if $a\geq 1$ then this is distinguished.

Theorem: (Weierstrass division) $g\in k\langle x_1,\dots,x_n\rangle$ x_n-distinguished of order s. Then $\forall f\in k\langle x_1,\dots,x_n\rangle$ $\exists! q\in k\langle x_1,\dots,x_n\rangle$, $\exists! r\in k\langle x_1,\dots,x_{n-1}\rangle[x_n]$ st
$$f = qg + r \quad \& \ \deg r < s.$$
Furthermore, $|f| = \max\{|q|.|g|, |r|\}$.

Homework 5: unicity in the above theorem & $|f|=\max\{|q||g|,|r|\}$.

Proof of the existence: $f=\sum f_i x_n^i$ $f_0=\sum_{i\leq t} f_i x_n^i$ write $f=f_0+\tilde{f}$ st $|\tilde{f}|<\varepsilon|f|$ and $|f_0|=|f|$. We may assume $|g|=1$.
Write g as,
$$g = g_0+\tilde{g} \text{ where } g_0=\sum_{i\leq s} g_i x_n^i \Rightarrow |\tilde{g}|<1 \ \& \ |g_0|=1.$$

$f_0 = q_0.g_0 + r_0$ division in $k\langle x_1,\dots,x_{n-1}\rangle[x_n]$ then $\deg r_0 < s$.
$$|f_0| = \max\{|q_0|.|g_0|, |r_0|\}$$
Because the highest value term of g_0 is in deg s. $\Rightarrow |f_0|\nless |q_0||g_0|$.
So $|f|=|f_0|=\max\{|q_0|,|r_0|\} \Rightarrow |f|\geq|q_0|$ and $|f|\geq|r_0|$.

$f = q_0.g + r_0 + (\tilde{f}-q_0.\tilde{g})$ let $\varepsilon=|\tilde{g}|$ now let $f_1 := \tilde{f}-q_0\tilde{g}$ then $|f_1|\leq\varepsilon|f|$

then repeat the argument to get $f_1 = q_1 g + r_1 + f_2$

$f = \sum f_i = (\sum q_i) g + (\sum r_i)$ convergent by the ε bounds.

WEEK 4:

Remark: $0 \neq f \in k\langle x\rangle \Rightarrow f$ is distinguished

Corollary: $k\langle x\rangle$ Euclidian w/ $N: k\langle x\rangle \setminus \{0\} \to \mathbb{Z}$, $\quad \Rightarrow k\langle x\rangle$ is PID.

$f \mapsto$ order to which f is distinguished.

Corollary 1: Suppose $k = \bar{k}$, $m \subseteq k\langle x\rangle$ max. ideal $\Leftrightarrow m = (x-\lambda)$ st $|\lambda| \leq 1$.

Corollary 2: $f \in k\langle x_1, \dots, x_n\rangle$ x_n-dist. $\Rightarrow \exists!$ monic $g \in k\langle x_1, \dots, x_{n-1}\rangle[x_n]$ & $u \in k\langle x_1, \dots, x_n\rangle^\times$ such that $f = g.u$ & $|g| = 1$.

Proof of 2: $s =$ order of f, by scaling we may assume $|f| = 1$.

$x_n^s = q.f + r$ & $|r|, |q| \leq 1 \Rightarrow x_n^s - r = q.f \Rightarrow$ letting $g = x_n^s - r$, $|g| \leq 1$ & g monic.

We need q invertible.

$\Rightarrow$ reduce to $R/m[x_1, \dots, x_n]$ $\quad \bar{g} = \bar{q}.\bar{f}$, x_n-degree of $\bar{f}$ & $\bar{g}$ are s and deg s coefficient is invertible.

$\Rightarrow$ after reduction constants $\Rightarrow \bar{q}$ constant $\Rightarrow q$ invertible.

Proof of 1: ① $m \subset k\langle x\rangle$ maximal $\Rightarrow m = (f)$ f irreducible $\Rightarrow f = u.g$ as above

$\Rightarrow m = (g)$ write $g = \prod_{i=1}^{r} (x - \lambda_i)$ st $|x - \lambda_i| \Rightarrow |\lambda_i| \leq 1$

One of $x - \lambda_i$ is not inv. $\Rightarrow (x - \lambda_i) = m$.

② $k\langle x\rangle \xrightarrow{ev_\lambda} k$ $\quad \sum a_i x^i \mapsto \sum a_i \lambda^i$ $\quad$ Claim: $\ker ev_\lambda = (x - \lambda)$. Use Euclidian division.

Corollary: for k not necessarily alg. closed, $\{$max ideals of $k\langle x\rangle\}$
← Nullstellensatz

$\uparrow$

$\{\ker ev_\lambda : \lambda \in \bar{k}, |\lambda| \leq 1\}$.

and $\lambda, \lambda' \in \bar{k}$ gives the same maximal ideal $\Leftrightarrow$ are Galois conjugate.

Corollary: $\overset{A}{k\langle x_1, \dots, x_n\rangle}$ is Noetherian.

Proof: $g \in I \subseteq A$ non-zero WMA g x_n-distinguished. Weiestrass division

$\Rightarrow A/g$ is a finite $k\langle x_1, \dots, x_{n-1}\rangle$-module by induction A/G is Noetherian module over $k\langle x_1, \dots, x_{n-1}\rangle \Rightarrow I/g$ is finitely gen. as a $k\langle x_1, \dots, x_{n-1}\rangle$ module and thus also as a $k\langle x_1, \dots, x_n\rangle$-module $\Rightarrow I$ finitely generated.

Homework 6: As in Rings & Modules show that Nullstellensatz holds for any n. [Hint: Noether norm.]

HOMEWORK 7

Example: $K = \mathbb{Q}_p$ $A = \mathbb{Z}_p$ or $K = \mathbb{Q}_p^{nr} = \mathbb{Q}_p\{$all prime to p roots of unity$\}$ étale $\mathbb{Q}_p$

$A = \{a \in \mathbb{Q}_p^{nr} \mid |a| \leq 1\}$ $\quad$ (a) $\Rightarrow \exists! \, |\cdot|_p$

$m = \{a \in \mathbb{Q}_p^{nr} \mid |a| < 1\} = p.A$ $\quad$ (b) $\Rightarrow \exists!$ extension of

$A/m = \bar{\mathbb{F}}_p$. $\quad$ $|\cdot|_p$ to $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}^p$.

$X = \mathrm{Spa}(K\langle x\rangle, A\langle x\rangle)$. There is a map $\mathrm{Spa}(K\langle x\rangle, A\langle x\rangle) \to \mathrm{Spec}\, K\langle x\rangle$

$|\cdot| \longmapsto \mathrm{Supp}|\cdot|$

$|\cdot|_x \longmapsto \ker ev_\lambda$ $\quad$ what are $|\cdot|_x$.

When $\mathrm{Supp}|\cdot|_x$ is not open $|\cdot|_x$ is called an analytic point. Note that $\ker ev_\lambda$ is not open.

For simplicity assume $\lambda \in K$,

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend left, "id"] & K\langle x\rangle \arrow[r, "ev_\lambda"] \arrow[rr, bend right, "\beta\ |\cdot|_x"'] & K \arrow[r, "|\cdot|"', "\alpha"] & \Gamma_{|\cdot|_x} \end{tikzcd}$$

$|\cdot|_x$ descends to a valuation

Diagram shows that $\beta^{-1}(\{\gamma < c\})$ open $\Rightarrow \alpha^{-1}(\{\gamma < c\})$ open $\Rightarrow$ descent $|\cdot|$ continuous.
$\Rightarrow |\cdot| = |\cdot|_p \Rightarrow |\sum a_i x^i| = |\sum a_i \lambda^i|_p$ Homework 8: The same thing happens when $\lambda \notin K$. Showing $|\cdot|$ cts. is the issue.

As soon as we show that $|A\langle x\rangle|_x = |ev_\lambda A\langle x\rangle|_p \leq 1$ (which is immediate) we have that $\mathbb{D} = \overline{B(0,1)} \subseteq \mathrm{Spa}(K\langle x\rangle, A\langle x\rangle)$ set theoretically.

Claim: $\mathbb{D} \longrightarrow \mathrm{Spa}(K\langle x\rangle, A\langle x\rangle)$ is a homeomorphism on its image.
$\lambda \longmapsto |\cdot|_{\lambda,p} := |\cdot| \circ ev_\lambda$

Proof: 1 $\{\lambda \in \mathbb{D} \mid |\lambda|_p < c\}$ open $R\left(\frac{x}{d}\right)$ with $|d| = c$ works.
2 Take $U = R\left(\frac{f_1 \cdots f_n}{g}\right)$ pick $\lambda_0 \in U$ want $\exists \varepsilon > 0$ st $|\lambda - \lambda_0|_p < \varepsilon \Rightarrow \lambda \in U$

Lemma: $f = \sum a_i x^i \in K\langle x\rangle$ $\lambda_0 \in \mathbb{D}$ st $|\sum a_i \lambda_0^i|_p$.
$\exists \varepsilon > 0$ st $|\lambda - \lambda_0|_p < \varepsilon \Rightarrow |\sum a_i \lambda^i|_p = |\sum a_i \lambda_0^i|_p$.

Proof: Fix n st $|a_i|_p < c$ for $i > n$. $\Rightarrow |\sum_{i=n+1}^{\infty} a_i \lambda^i|_p < c$. So enough to show $\exists 1 \gg \varepsilon > 0$ st $|\lambda - \lambda_0| < \varepsilon \Rightarrow |\sum_{i=0}^{n} a_i \lambda^i|_p = |\sum_{i=0}^{n} a_i \lambda_0^i|_p$.

Consider $|\sum_{i=0}^{n} a_i(\lambda^i - \lambda_0^i)|_p \leq \max_i |a_i|_p |\lambda - \lambda_0|_p$ choose ε st this is smaller than c and we are done.